Appendix for "Network localization is unalterable by infections in bursts"

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Appendix A
The coefficient a_{max}

If the adjacency matrix of the network is A, the largest eigenvalue of A is λ_1, the normalized principal eigenvalue of A is $x = [x_1, \ldots, x_N]^T$, and the effective infection rate is $\tau = \beta/\delta$ with infection rate β and curing rate δ, then the epidemic threshold [1, Theorem 1] of the bursty SIS model is $\tau_c^{(B)} = \frac{1}{\ln(\lambda_1+1)}$ and the following Theorem holds.

Theorem 1. *For the bursty SIS process with effective infection rate τ above the threshold $\tilde{\tau} \triangleq \frac{\tau}{\tau_c^{(B)}} - 1 > 0$, the maximum steady-state prevalence is $y_\infty^+(\tilde{\tau}) = a_{max}\tilde{\tau} + o(\tilde{\tau})$ with*

$$a_{max} = \frac{2}{N} \frac{(\lambda_1+1)\ln(\lambda_1+1)\sum_{i=1}^N x_i}{\lambda_1 \sum_{i=1}^N x_i^3}$$

and the minimum prevalence is $y_\infty^-(\tilde{\tau}) = a_{min}\tilde{\tau} + o(\tilde{\tau})$ with $a_{min} = a_{max}/(\lambda_1+1)$.

To prove Theorem 1, we first prove the following Lemma.

Lemma 2.

$$\sum_{i=1}^N x_i \sum_{\{j,k\in\mathcal{N}_i | j<k\}} x_j x_k + \lambda_1 \sum_{i=1}^N x_i^3 = \frac{1}{2}\lambda_1(\lambda_1+1)\sum_{i=1}^N x_i^3$$

where $\mathcal{N}_i$ denotes the set of neighbors of node i.

Proof of Lemma 2. For the first term on the left-hand side, we have

$$\begin{aligned}\sum_{i=1}^N x_i \sum_{\{j,k\in\mathcal{N}_i | j<k\}} x_j x_k &= \frac{1}{2}\sum_{i=1}^N x_i \sum_{j\in\mathcal{N}_i} x_j \left(\sum_{k\in\mathcal{N}_i} x_k - x_j\right) \\ &= \frac{1}{2}\sum_{i=1}^N x_i \sum_{j\in\mathcal{N}_i} x_j \sum_{k\in\mathcal{N}_i} x_k \\ &\quad -\frac{1}{2}\sum_{i=1}^N x_i \sum_{j\in\mathcal{N}_i} x_j^2 \end{aligned} \quad (1)$$

Since $\sum_{j\in\mathcal{N}_i} x_j = \lambda_1 x_i$, the first term of (1) is $\frac{1}{2}\lambda_1^2 \sum_{i=1}^N x_i^3$. We consider the second term of (1)

$$\begin{aligned}-\frac{1}{2}\sum_{i=1}^N x_i \sum_{j\in\mathcal{N}_i} x_j^2 &= -\frac{1}{2}\sum_{\forall \text{link}(i,j)} (x_i^2 x_j + x_i x_j^2) \\ &= -\frac{1}{2}\sum_{i=1}^N x_i^2 \sum_{j\in\mathcal{N}_i} x_j \\ &= -\frac{1}{2}\lambda_1 \sum_{i=1}^N x_i^3\end{aligned}$$

Thus, the left-hand side equals $\frac{1}{2}\lambda_1(\lambda_1+1)\sum_{i=1}^N x_i^3$. □

Proof of Theorem 1. The mean-field governing equations of the bursty SIS process are [1],

$$v_i\left(\frac{n+1}{\beta}\right) = \lim_{t^*\to 1/\beta}\left(\left[1 - v_i\left(t^* + \frac{n}{\beta}\right)\right]\left\{1 - \prod_{j\in\mathcal{N}_i}\left[1 - v_j\left(t^* + \frac{n}{\beta}\right)\right]\right\} + v_i\left(t^* + \frac{n}{\beta}\right)\right) \quad (2)$$

and

$$\frac{\mathrm{d}v_i\left(\frac{n}{\beta} + t^*\right)}{\mathrm{d}t^*} = -\delta v_i\left(\frac{n}{\beta} + t^*\right) \quad (3)$$

where $v_i(t)$ is the infection probability of node i at time t, $t^* \in [0, 1/\beta)$ is the length of the time passed after the nearest burst, and $\mathcal{N}_i$ denotes the set of neighbor nodes of node i. The solution of Eq. (3) is

$$v_i\left(\frac{n}{\beta} + t^*\right) = v_i\left(\frac{n}{\beta}\right)\mathrm{e}^{-\delta t^*} \quad (4)$$

Substituting (4) at $t^* \to 1/\beta$, i.e. $\lim_{t^*\to 1/\beta} v_i(n/\beta + t^*) = v_i(n/\beta)\exp(-1/\tau)$, into Eq. (2), we obtain the following recursion of the infection probability of each node at $t^* = 0$ just after each burst,

$$v_i\left(\frac{n+1}{\beta}\right) = \left(1 - v_i\left(\frac{n}{\beta}\right)\mathrm{e}^{-1/\tau}\right)\left(1 - \prod_{j\in\mathcal{N}_i}\left(1 - v_i\left(\frac{n}{\beta}\right)\mathrm{e}^{-1/\tau}\right)\right) + v_j\left(\frac{n}{\beta}\right)\mathrm{e}^{-1/\tau} \quad (5)$$

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Equation (6) is the discrete-time SIS equation with infection probability $\tilde{\beta} = \mathrm{e}^{-1/\tau}$ and curing probability $\tilde{\delta} = 1 - \mathrm{e}^{-1/\tau}$. We rewrite Eq. (5) as,

$$p_i[n+1] = \left(1 - (1-\tilde{\delta})p_i[n]\right)\left(1 - \prod_{j\in\mathcal{N}_i}\left(1 - \tilde{\beta}p_j[n]\right)\right) + p_j[n](1-\tilde{\delta})$$

where $p_i[n] \triangleq v_i(n/\beta)$. In the steady state, $\lim_{n\to\infty} p_i[n+1] = \lim_{n\to\infty} p_i[n] = p_{i\infty}$ for $1 \le i \le N$, and we have,

$$\tilde{\delta}p_{i\infty} = \left(1 - (1-\tilde{\delta})p_{i\infty}\right)\left(1 - \prod_{j\in\mathcal{N}_i}\left(1 - \tilde{\beta}p_{j\infty}\right)\right) \tag{6}$$

In the steady state, the discrete-time SIS infection probability vector $p_\infty \triangleq [p_{1\infty}, \ldots, p_{N\infty}]$ approaches an eigenvector of the adjacency matrix A corresponding to the largest eigenvalue λ_1 when $\tilde{\beta}/\tilde{\delta} \downarrow 1/\lambda_1$. Thus, we can assume $p_\infty = ax + o(a)q$, where q is a vector orthogonal to x and with finite components.

Substituting $p_\infty = ax + o(a)q$ into (6), we obtain,

$$\begin{aligned}\tilde{\delta}ax_i + \tilde{\delta}o(a)q_i =& \tilde{\beta}a\sum_{j\in\mathcal{N}_i} x_j + \tilde{\beta}o(a)\sum_{j\in\mathcal{N}_j} q_j - \\ & a^2\tilde{\beta}^2\sum_{\{j,k\in\mathcal{N}_i|j<k\}} x_jx_k - \\ & \tilde{\beta}(1-\tilde{\delta})a^2x_i\sum_{j\in\mathcal{N}_i} x_j + o(a^2)\end{aligned} \tag{7}$$

where the eigenvalue equation indicates that $\sum_{j\in\mathcal{N}_i} x_j = \lambda_1 x_i$.

In vector form, (7) is,

$$\begin{aligned}\tilde{\delta}ax + \tilde{\delta}o(a)q =& \tilde{\beta}aAx + \tilde{\beta}o(a)Aq - \\ & a^2\tilde{\beta}^2\mathrm{vec}\left(\sum_{\{j,k\in\mathcal{N}_i|j<k\}} x_jx_k\right) - \\ & \tilde{\beta}(1-\tilde{\delta})a^2\mathrm{vec}\left(\lambda_1 x_i^2\right) + o(a^2)h\end{aligned} \tag{8}$$

where the vector $\mathrm{vec}(z_i) \triangleq [z_1, \ldots, z_N]^T$. Divide both sides of (8) by $a\tilde{\beta}$ and recall that $Ax = \lambda_1 x$, and we have

$$\begin{aligned}\frac{\tilde{\delta}}{\tilde{\beta}}x + \frac{\tilde{\delta}}{\tilde{\beta}}\frac{o(a)}{a}q =& \lambda_1 x + \frac{o(a)}{a}Aq - \\ & a\tilde{\beta}\mathrm{vec}\left(\sum_{\{j,k\in\mathcal{N}_i|j<k\}} x_jx_k\right) - \\ & a(1-\tilde{\delta})\mathrm{vec}\left(\lambda_1 x_i^2\right) + \frac{o(a^2)}{a}h\end{aligned} \tag{9}$$

Rearranging (9), we obtain

$$\begin{aligned}\left(\lambda_1 - \frac{\tilde{\delta}}{\tilde{\beta}}\right)x - \frac{\tilde{\delta}}{\tilde{\beta}}\frac{o(a)}{a}q =& -\frac{o(a)}{a}Aq + \\ & a\Bigg[\tilde{\beta}\mathrm{vec}\left(\sum_{\{j,k\in\mathcal{N}_i|j<k\}} x_jx_k\right) + \\ & (1-\tilde{\delta})\mathrm{vec}\left(\lambda_1 x_i^2\right)\Bigg] + \frac{o(a^2)}{a}h\end{aligned} \tag{10}$$

Since $a \to 0$ as $(\lambda_1 - \tilde{\delta}/\tilde{\beta}) \to 0$, we assume

$$a = a_1(\lambda_1 - \tilde{\delta}/\tilde{\beta}) + o(\lambda_1 - \tilde{\delta}/\tilde{\beta}) \tag{11}$$

and substitute a into (10),

$$\begin{aligned}\left(\lambda_1 - \frac{\tilde{\delta}}{\tilde{\beta}}\right)x - \frac{\tilde{\delta}}{\tilde{\beta}}\frac{o(a)}{a}q =& -\frac{o(a)}{a}Aq + \\ & a_1\left(\lambda_1 - \frac{\tilde{\delta}}{\tilde{\beta}}\right)d(\tilde{\beta},\tilde{\delta}) + \\ & o(a)d(\tilde{\beta},\tilde{\delta}) + \frac{o(a^2)}{a}h\end{aligned} \tag{12}$$

where

$$d(\tilde{\beta},\tilde{\delta}) = \left(\tilde{\beta}\mathrm{vec}\left(\sum_{\{j,k\in\mathcal{N}_i|j<k\}} x_jx_k\right) + (1-\tilde{\delta})\mathrm{vec}\left(\lambda_1 x_i^2\right)\right).$$

We divide both side by $\lambda_1 - \tilde{\delta}/\tilde{\beta}$,

$$\begin{aligned}x - \frac{\tilde{\delta}}{\tilde{\beta}}\frac{o(a)}{a}\frac{1}{\left(\lambda_1 - \frac{\tilde{\delta}}{\tilde{\beta}}\right)}q =& -\frac{o(a)}{a}\frac{1}{\left(\lambda_1 - \frac{\tilde{\delta}}{\tilde{\beta}}\right)}Aq + \\ & a_1 d(\tilde{\beta},\tilde{\delta}) + \frac{o(a)}{\left(\lambda_1 - \frac{\tilde{\delta}}{\tilde{\beta}}\right)}d(\tilde{\beta},\tilde{\delta}) + \\ & \frac{o(a^2)}{a\left(\lambda_1 - \frac{\tilde{\delta}}{\tilde{\beta}}\right)}h\end{aligned} \tag{13}$$

By taking the scalar product with x on both sides of Eq. (13) and recalling that the vector q is orthogonal to the eigenvector x, we obtain

$$1 = a_1 d(\tilde{\beta},\tilde{\delta})\cdot x + \frac{o(a)}{\left(\lambda_1 - \frac{\tilde{\delta}}{\tilde{\beta}}\right)}d(\tilde{\beta},\tilde{\delta})\cdot x + \frac{o(a^2)}{a\left(\lambda_1 - \frac{\tilde{\delta}}{\tilde{\beta}}\right)}h\cdot x \tag{14}$$

When $a \to 0$, Eq. (14) becomes

$$1 = a_1 d\left(\frac{1}{\lambda_1}\tilde{\delta}, \tilde{\delta}\right)\cdot x \tag{15}$$

In the bursty SIS case where $\lim_{\tau\downarrow\tau_c^{(B)}} \tilde{\delta} = \frac{\lambda_1}{\lambda_1+1}$, Eq. (15) reads

$$1 = a_1 d\left(\frac{1}{\lambda_1+1}, \frac{\lambda_1}{\lambda_1+1}\right)\cdot x$$

Thus,

$$a_1 = \frac{\lambda_1 + 1}{\sum_{i=1}^N x_i\sum_{\{j,k\in\mathcal{N}_i|j<k\}} x_jx_k + \lambda_1\sum_{i=1}^N x_i^3}$$

Using Lemma 2, a_1 becomes

$$a_1 = \frac{2}{\lambda_1\sum_{i=1}^N x_i^3}$$

We assume $a = a_2\epsilon + o(\epsilon)$ where $\epsilon = \tau - \tau_c^{(B)} = \tau - \frac{1}{\ln(\lambda_1+1)}$ and we may verify that

$$\begin{aligned}\left.\frac{\mathrm{d}(\lambda_1 - \tilde{\delta}/\tilde{\beta})}{\mathrm{d}\epsilon}\right|_{\epsilon=0} &= \left.\frac{\mathrm{d}(\lambda_1 + 1 - \mathrm{e}^{1/\tau})}{\mathrm{d}\epsilon}\right|_{\epsilon=0} \\ &= (\lambda_1+1)\ln^2(\lambda_1+1)\end{aligned}$$

then we obtain

$$a_2 = a_1(\lambda_1+1)\ln^2(\lambda_1+1) = \frac{2(\lambda_1+1)\ln^2(\lambda_1+1)}{\lambda_1\sum_{i=1}^N x_i^3}$$

Thus, the maximum prevalence is $\frac{a_2\sum_{i=1}^N x_i}{N}\left(\tau-\tau_c^{(B)}\right)+o(\tau-\tau_c^{(B)})$.

After normalizing the effective infection rate by $\tau/\tau_c^{(B)}$ and defining $\tilde{\tau}=\tau/\tau_c^{(B)}-1$, we finally find the maximum prevalence as

$$\begin{aligned} y_\infty^+(\tilde{\tau}) &= \frac{a_2\tau_c^{(B)}\sum_{i=1}^N x_i}{N}\tilde{\tau}+o(\tilde{\tau}) \\ &= \frac{2(\lambda_1+1)\ln(\lambda_1+1)\sum_{i=1}^N x_i}{N\lambda_1\sum_{i=1}^N x_i^3}\tilde{\tau}+o(\tilde{\tau}) \quad (16)\end{aligned}$$

For general t^*, the prevalence is $\exp(-\delta t^*)y_\infty^+(\tilde{\tau})$ and then the minimum prevalence is $y_\infty^-(\tilde{\tau}) = y_\infty^+(\tilde{\tau})/(\lambda_1+1)$ as $t^*\to 1/\beta$. □

Appendix B
The bounds of a

By the Perron-Frobenius theorem, every component of the principal eigenvector is positive. The lower bound of a is derived follows.

$$a=\frac{\sum_{i=1}^N x_i}{N\sum_{i=1}^N x_i^3}\geq\frac{N\min_i x_i}{N\max_i x_i\sum_{j=1}^N x_j^2}=\frac{\min_i x_i}{\max_i x_i}$$

For the upper bound, using the CauchySchwarz inequality $(\sum_{i=1}^N x_i)^2\leq N\sum_{i=1}^N = N$, we obtain

$$a=\frac{\sum_{i=1}^N x_i}{N\sum_{i=1}^N x_i^3}\leq\frac{\sqrt{N}}{N\min_i x_i}=\frac{1}{\sqrt{N}\min_i x_i}$$

The bound is tight when the network is a regular graph.

Appendix C
The coefficients of star graphs

We may verify that the largest eigenvalue of the star graph is $\sqrt{N-1}$ and the principle eigenvector is $x = [\frac{1}{\sqrt{2}},\ldots,\frac{1}{\sqrt{2(N-1)}}]^T$. We have following results

$$\begin{aligned} a=\frac{\sum_{i=1}^N x_i}{N\sum_{i=1}^N x_i^3} &= \frac{1}{\sqrt{N}}+o(\frac{1}{\sqrt{N}}) \\ a_{\max} &= \frac{2}{N}\frac{(\lambda_1+1)\ln(\lambda_1+1)\sum_{i=1}^N x_i}{\lambda_1\sum_{i=1}^N x_i^3} \\ &= \frac{\ln(\sqrt{N})}{\sqrt{N}}+o(N^{-\frac{1}{2}}\ln N) \\ a_{\min} &= \frac{2}{N}\frac{\ln(\lambda_1+1)\sum_{i=1}^N x_i}{\lambda_1\sum_{i=1}^N x_i^3} \\ &= \frac{\ln(\sqrt{N})}{N}+o(N^{-1}\ln N)\end{aligned}$$

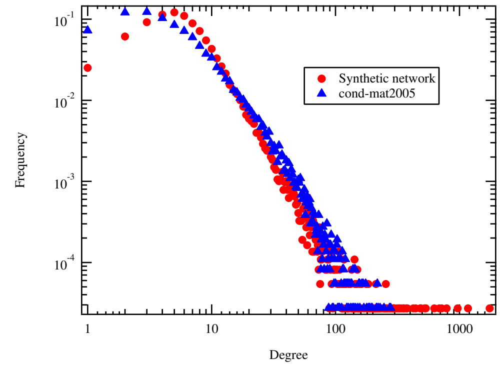

Fig. 1. Cond-mat2005:: Collaboration network of scientists posting preprints on the condensed matter archive at arXiv, 1995-1999.

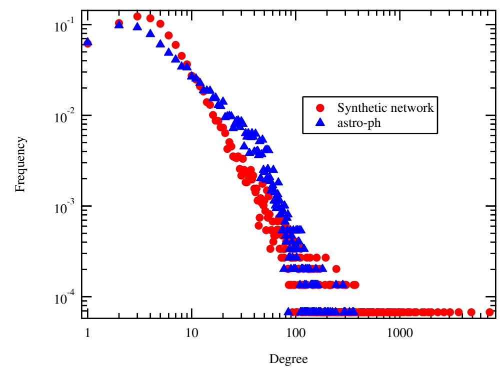

Fig. 2. Astro-ph: Network of co-authorship between scientists posting preprints on the Astrophysics E-Print Archive between Jan 1, 1995 and December 31, 1999.

Appendix D
The coefficients of d-regular graphs

For regular graph, the principal eigenvector is $x=\frac{1}{\sqrt{N}}u$ where u is all-one vector and the largest eigenvalue is d. We may verify that

$$\begin{aligned} a=\frac{\sum_{i=1}^N x_i}{N\sum_{i=1}^N x_i^3} &= 1 \\ a_{\max}=\frac{2}{N}\frac{(\lambda_1+1)\ln(\lambda_1+1)\sum_{i=1}^N x_i}{\lambda_1\sum_{i=1}^N x_i^3} &= 2\left(1+\frac{1}{d}\right)\ln(d+1) \\ a_{\min}=\frac{2}{N}\frac{\ln(\lambda_1+1)\sum_{i=1}^N x_i}{\lambda_1\sum_{i=1}^N x_i^3} &= \frac{2\ln(d+1)}{d}\end{aligned}$$

Appendix E
Real networks

The parameters of the real and synthetic networks are listed in Table 1. The degree distributions are plotted in Fig. 1 to Fig. 7.

References

[1] Q. Liu and P. Van Mieghem, "Burst of virus infection and a possibly largest epidemic threshold of non-markovian susceptible-infected-susceptible processes on networks," *Physical Review E*, vol. 97, no. 2, p. 022309, 2018.
[2] M. E. Newman, "The structure of scientific collaboration networks," *Proceedings of the national academy of sciences*, vol. 98, no. 2, pp. 404–409, 2001.

Networks	Parameters of real networks	Parameters of corresponding synthetic networks
Cond-mat 2005 [2]	$N = 36458$, $d_{av} = 9.4210$, $a_{\max} = 0.1199$	$N = 36811$, $d_{av} = 9.4483$, $a_{\max} = 0.1301$
astro-ph [2]	$N = 14845$, $d_{av} = 16.1202$, $a_{\max} = 0.3024$	$N = 14766$, $d_{av} = 16.2536$, $a_{\max} = 0.5352$
Internet [3]	$N = 22963$, $d_{av} = 4.2186$, $a_{\max} = 0.2155$	$N = 22354$, $d_{av} = 4.2804$, $a_{\max} = 0.1903$
hep-th [2]	$N = 5835$, $d_{av} = 4.7352$, $a_{\max} = 0.0218$	$N = 5944$, $d_{av} = 4.5855$, $a_{\max} = 0.1063$
Email-URV [4]	$N = 1133$, $d_{av} = 9.6222$, $a_{\max} = 1.3713$	$N = 1178$, $d_{av} = 9.6774$, $a_{\max} = 0.9539$
PGP [5]	$N = 10680$, $d_{av} = 4.5536$, $a_{\max} = 0.0789$	$N = 10986$, $d_{av} = 4.4773$, $a_{\max} = 0.2104$
Email-Enron [6]	$N = 33696$, $d_{av} = 10.7319$, $a_{\max} = 0.3037$	$N = 33632$, $d_{av} = 10.8451$, $a_{\max} = 0.3053$

TABLE 1
The parameters of real networks and the corresponding synthetic networks. Only the largest connected components are preserved and all the networks are connected.

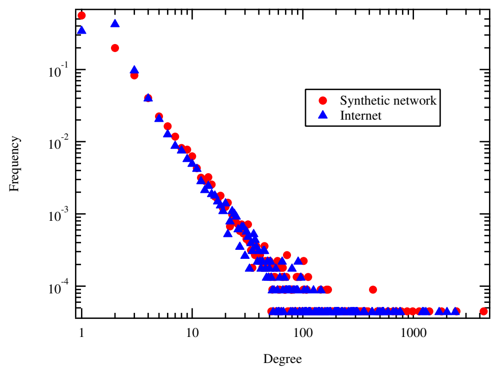

Fig. 3. Internet: a symmetrized snapshot of the structure of the Internet at the level of autonomous systems, reconstructed from BGP tables posted by the University of Oregon Route Views Project.

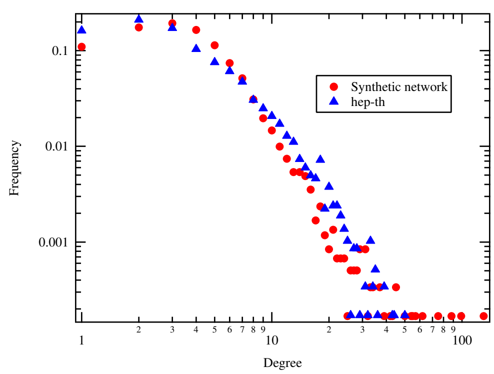

Fig. 4. Hep-th: Network of co-authorship between scientists posting preprints on the High-Energy Theory arXiv between Jan 1, 1995 and December 31, 1999.

[3] M. Newman, “Mark Newman’s network data,” http://www-personal.umich.edu/ mejn/netdata/, Apr. 2013.
[4] R. Guimera, L. Danon, A. Diaz-Guilera, F. Giralt, and A. Arenas, “Self-similar community structure in a network of human interactions,” *Physical review E*, vol. 68, no. 6, p. 065103, 2003.
[5] M. Boguná, R. Pastor-Satorras, A. Díaz-Guilera, and A. Arenas, “Models of social networks based on social distance attachment,” *Physical review E*, vol. 70, no. 5, p. 056122, 2004.
[6] J. Leskovec and A. Krevl, “SNAP Datasets: Stanford large network dataset collection,” http://snap.stanford.edu/data, Jun. 2014.

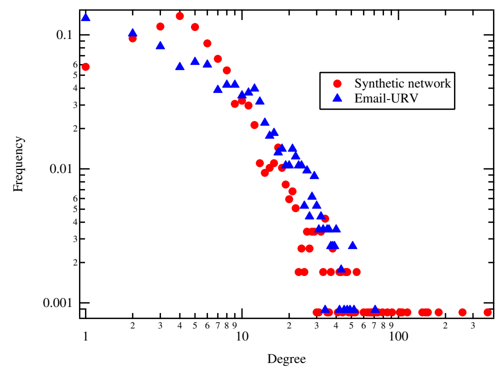

Fig. 5. Email-URV: Network of E-mail interchanges between members of the Univeristy Rovira i Virgili, Tarragona.

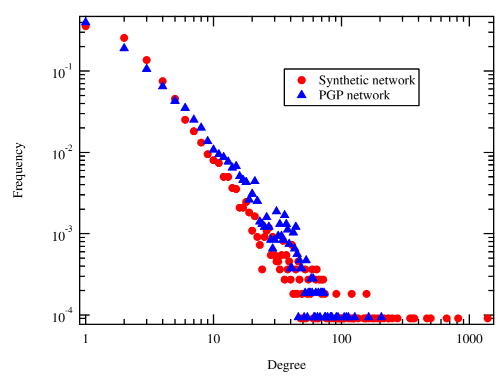

Fig. 6. PGP: Network of users of the Pretty-Good-Privacy algorithm for secure information interchange.

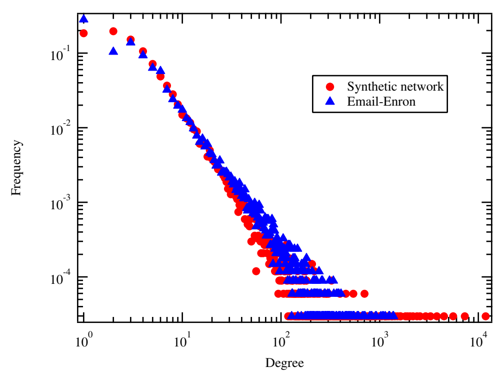

Fig. 7. Email-Enron: Enron email communication network.